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A SET OF FIVE POSTULATES FOR BOOLEAN ALGEBRAS IN TERMS OF THE OPERATION "EXCEPTION"

BY

J. S. TAYLOR

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INTRODUCTION

There are three binary operations between classes which have come into general use in Boolean algebras. These three are "logical addition," "rejection," and "exception," and are expressed respectively by the symbols "$+$," "$|$," and "$-$." Simple and elegant sets of postulates already exist for the logic of classes in terms of "logical addition,"[1] and in terms of "rejection."[2] The set of postulates by B. A. Bernstein,[3] however, to whom the third operation is due, is somewhat involved and it is therefore the purpose of this paper to present a comparatively simp'e set in terms of "exception."

I

A SET OF FIVE POSTULATES FOR BOOLEAN ALGEBRAS IN TERMS OF THE OPERATION "EXCEPTION"

Let us take as undefined ideas a *class* K of *elements* $a, b, c, \ldots\ldots$ and an operation "$-$," $a-b$ reading "a except b." Then the logic of classes may be defined as a system $\Sigma\ (K,-)$ which satisfies the following five postulates:

Postulates

I. K contains at least two distinct elements.

II. If a and b are elements of K, $a-b$ is an element of K.

III. If a, b, and the combinations indicated are elements of K,

$$a-(b-b)=a.$$

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[1] E. V. HUNTINGTON, "Sets of Independent Postulates for the Algebra of Logic," *Transactions of the American Mathematical Society*, Vol. V (1904), pp. 288–309.

[2] B. A. BERNSTEIN, "A Set of Four Independent Postulates for Boolean Algebras," *Transactions of the American Mathematical Society*, vol. XVII, pp. 50–52.

[3] B. A. BERNSTEIN, "A Complete Set of Postulates for the Logic of Classes in Terms of the Operation 'Exception', and a Proof of the Independence of a Set of Postulates Due to Del Re," *Univ. Calif. Publ. Math.*, vol. I, pp. 87–96 (May 15, 1914).

IV. There exists a unique element *1* in K such that, if a, b, and the combinations indicated are elements of K,

$$a-(1-b)=b-(1-a).$$

Definition 1. $a^{\cdot}=1-a$.

V. If IV holds, and if a, b, c, and the combinations indicated are elements of K,

$$a-(b-c)=[(a-b)^{\cdot}-(a-c^{\cdot})]^{\cdot}.$$

Definition 2. $a^{\cdot\cdot}=(a^{\cdot})^{\cdot}$.

Theorems

Theorem 1. $a^{\cdot\cdot}=a$

Proof. $a-(1-1)=1-(1-a)$ by IV.

but $a-(1-1)=a$ by III.

and $1-(1-a)=1-a^{\cdot}$ by Def. 1.

$=(a^{\cdot})^{\cdot}$ by Def. 1.

$=a^{\cdot\cdot}$ by Def. 2.

Theorem 2. $a^{\cdot}$ is unique (for any a in K). by IV, II.

Theorem 3. $a-b^{\cdot}=b-a^{\cdot}$ by IV, Def. 1.

Corollary 1. $a^{\cdot}-b=b^{\cdot}-a$

Corollary 2. $a-b=b^{\cdot}-a^{\cdot}$

Theorem 4. $(a-a)^{\cdot}=(b-b)^{\cdot}$

Proof. $(a-a)^{\cdot}-(b-b)=(a-a)^{\cdot}$ by III.

and $(a-a)^{\cdot}-(b-b)=(b-b)^{\cdot}-(a-a)$ by Th. 3, Cor. 1.

$=(b-b)^{\cdot}$ by III.

Theorem 5. $1=(e-e)^{\cdot}$.

Proof. First, $(e-e)^{\cdot}$ is unique. by II; Th's 2 and 4.

Secondly, $(e-e)^{\cdot}$ satisfies the equation of IV, in other words,

$$a-[(e-e)^{\cdot}-b]=b-[(e-e)^{\cdot}-a]$$

for, $a-[(e-e)^{\cdot}-b]=a-[b^{\cdot}-(e-e)]$ by Th. 3, Cor. 1.

$=a-b^{\cdot}$ by III.

and $b-[(e-e)^{\cdot}-a]=b-[a^{\cdot}-(e-e)]$ by Th. 3, Cor. 1.

$=b-a^{\cdot}$ by III.

but $a-b^{\cdot}=b-a^{\cdot}$ by Th. 3.

Theorem 6. $a-a^{\cdot}=a$

Proof. Set $b=c=a^{\cdot}$ and $a=a$ in V.

The left then becomes $a-(a^{\cdot}-a^{\cdot})=a$ by III.

The right becomes $[(a-a^{\cdot})^{\cdot}-(a-a^{\cdot\cdot})]^{\cdot}=[(a-a^{\cdot})^{\cdot}-(a-a)]^{\cdot}$ by Th. 1.

$=[(a-a^{\cdot})^{\cdot}]^{\cdot}$ by III.

$=a-a^{\cdot}$ by Th. 1.

Corollary. $a^{\cdot}-a=a^{\cdot}$

Theorem 7. $a-(b-c)=[(b^{\cdot}-a^{\cdot})^{\cdot}-(c-a^{\cdot})]^{\cdot}$ by V; Th. 3; Th. 3, Cor. 2.

Corollary. $a^{\cdot\cdot}-(b^{\cdot\cdot}-c)=[(b^{\cdot}-a^{\cdot})^{\cdot}-(c^{\cdot\cdot}-a^{\cdot})]^{\cdot}$ by Th. 1.

Theorem 8. $(b^{\cdot}-a)^{\cdot}-(b-a)=a$

Proof. Set $a=a^{\cdot}$, $c=b$ in Th. 7, which then becomes

$$a^{\cdot}-(b-b)=[(b^{\cdot}-a^{\cdot\cdot})^{\cdot}-(b-a^{\cdot\cdot})]^{\cdot}$$

but $a^{\cdot}-(b-b)=a^{\cdot}$ by III.

and $[(b^{\cdot}-a^{\cdot\cdot})^{\cdot}-(b-a^{\cdot\cdot})]^{\cdot}=[(b^{\cdot}-a)^{\cdot}-(b-a)]^{\cdot}$ by Th. 1.

hence $[(b^{\cdot}-a)^{\cdot}-(b-a)]^{\cdot}=a^{\cdot}$

hence $[(b^{\cdot}-a)^{\cdot}-(b-a)]^{\cdot\cdot}=a^{\cdot\cdot}$ by Th. 2.

and $(b^{\cdot}-a)^{\cdot}-(b-a)=a$ by Th. 1.

Corollary. $(b^{\cdot}-a)^{\cdot}-(b^{\cdot\cdot}-a)=a$

Definition 3. $a \mid b=a^{\cdot}-b$

Theorem 9. $a \mid a=a^{\cdot}$ by Th. 6, Cor. and Def. 3.

Definition 4. $a'=a \mid a$

Theorem 10. $a'=a^{\cdot}$ by Def. 4, Th. 9.

Theorem 11. $(b \mid a) \mid (b' \mid a)=a$

Proof. $(b^{\cdot}-a)^{\cdot}-(b^{\cdot\cdot}-a)=a$ by Th. 8, Cor.

but $(b^{\cdot}-a)^{\cdot}-(b^{\cdot\cdot}-a)=(b \mid a)^{\cdot}-(b^{\cdot} \mid a)$ by Def. 3.

$=(b \mid a) \mid (b' \mid a)$ by Def. 3; Th. 10.

Theorem 12. $a' \mid (b' \mid c)=[(b \mid a') \mid (c' \mid a')]'$

Proof. $a^{\cdot\cdot}-(b^{\cdot\cdot}-c)=[(b^{\cdot}-a^{\cdot})^{\cdot}-(c^{\cdot\cdot}-a^{\cdot})]^{\cdot}$ by Th. 7, Cor.

but $a^{\cdot\cdot}-(b^{\cdot\cdot}-c)=a^{\cdot\cdot}-(b^{\cdot} \mid c)$ by Def. 3.

$=a^{\cdot} \mid (b^{\cdot} \mid c)$ by Def. 3.

$=a' \mid (b' \mid c)$ by Th. 10.

and $[(b^{\cdot}-a^{\cdot})^{\cdot}-(c^{\cdot\cdot}-a^{\cdot})]^{\cdot}=[(b \mid a^{\cdot})^{\cdot}-(c^{\cdot} \mid a^{\cdot})]^{\cdot}$ by Def. 3.

$=[(b \mid a^{\cdot}) \mid (c^{\cdot} \mid a^{\cdot})]^{\cdot}$ by Def. 3.

$=[(b \mid a') \mid (c' \mid a')]'$ by Th. 10.

Sufficiency

That postulates I–V are sufficient is now evident. In the light of Definition 3 postulates I and II give P_1 and P_2 of B. A. Bernstein's set of four postulates in terms of "rejection" referred to in an earlier part of this paper; while P_3 and P_4 of that set are here exhibited as theorems 11 and 12. That postulates I–V may likewise be derived from P_1-P of Bernstein's set is also easily shown. Thus the two sets of postulates are equivalent.

Consistency

The consistency of the set of postulates I–V is demonstrated by the following system composed of two distinct elements e_1 and e_2 which satisfies all five postulates. As in succeeding examples, e_i-e_j will be given by means of a table; so that if, as

in the present instance, $e_1 - e_1 = e_2$, $e_1 - e_2 = e_1$, $e_2 - e_1 = e_2$, and $e_2 - e_2 = e_2$, this will be stated in the form:

$-$	e_1	e_2
e_1	e_2	e_1
e_2	e_2	e_2

That this system Σ satisfies all the postulates the reader may verify without difficulty.

Independence

The independence of the five postulates is demonstrated by exhibiting five systems each satisfying all but one of the postulates, the unsatisfied postulate being I – V in turn. The system Σ failing to satisfy the i^{th} postulate will be designated Σ^{-i}. $e_i - e_j = x$ means that $e_i - e_j$ does not give an element belonging to K.

Σ^{-1}; K a class of one element e_1, with $e_1 - e_1 = e_1$.

Σ^{-2}; K a class of two distinct elements e_1 and e_2, with $e_i - e_j$ defined by the accompanying table:

$-$	e_1	e_2
e_1	e_1	e_1
e_2	e_2	x

Σ^{-3}; K a class of three distinct elements, with $e_i - e_j$ defined by the accompanying table:

$-$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_2	e_1	e_2
e_3	e_1	e_1	e_1

III fails for $a = e_3$.

IV holds, for e_2, and e_2 only, satisfies the conditions imposed on 1.

V holds. It holds obviously for a, b, and c limited to e_1 and e_2, for that part of K is identical with the system used to demonstrate the consistency of the postulates with e_1 and e_2 simply interchanged. The other possibilities may be disposed of as follows:

(1) $a = e_1$ or e_3, $b = e_i$, $c = e_j$ $(i, j = 1, 2, 3)$.

$$e_{1 \text{ or } 3} - (e_i - e_j) = e_1, \quad [(e_{1 \text{ or } 3} - e_i)^{\cdot} - (e_{1 \text{ or } 3} - e_j^{\cdot})]^{\cdot} = [e_2 - e_1]^{\cdot} = e_1$$

(2) For $a = e_2$ we have the following as yet undisposed of cases:

$$e_2 - (e_3 - e_i) = e_2, \quad [(e_2 - e_3)^{\cdot} - (e_2 - e_i^{\cdot})]^{\cdot} = [e_1 - e_j]^{\cdot} = e_2$$

$$e_2 - (e_1 - e_3) = e_2, \quad [(e_2 - e_1)^{\cdot} - (e_2 - e_3^{\cdot})]^{\cdot} = [e_1 - e_1]^{\cdot} = e_2$$

$$e_2 - (e_2 - e_3) = e_1, \quad [(e_2 - e_2)^{\cdot} - (e_2 - e_3^{\cdot})]^{\cdot} = [e_2 - e_1]^{\cdot} = e_1$$

Σ^{-4}; K a class of two distinct elements e_1 and e_2 with $e_i - e_j$ defined by the table:

$-$	e_1	e_2
e_1	e_1	e_1
e_2	e_2	e_2

Neither e_1 nor e_2 satisfies the conditions imposed on 1 by IV.

V is satisfied vacuously.

Σ^{-5}; K a class of three distinct elements with $e_i - e_j$ defined by the table:

$-$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_2	e_1	e_3
e_3	e_3	e_1	e_1

III is obviously satisfied.

IV is satisfied, for e_2 satisfies the conditions imposed on 1.

V fails for $a = b = c = e_3$.

II

COMPLETE EXISTENTIAL THEORY

As has already been shown, postulates I–V are independent in the ordinary sense that no one of the postulates is implied by the other four. Professor E. H. Moore,[4] however, has suggested the question in connection with sets of postulates of determining not only the implicational relations existing among the postulates as they stand, but also all the implicational relations which exist among properties defined either by the postulates themselves or by the negatives of the postulates. A set of postulates is said to be *completely independent* if, and only if, no such implicational relations exist. For example, I have shown in an earlier paper[5] that while Bernstein's set of four postulates in terms of "rejection" already referred to are independent in the ordinary sense, they are not completely independent, since the negative of the first postulate implies the third and fourth.

Any system Σ $(K, -)$ of the type prescribed earlier in this paper has with respect to the five postulates there stated one of the $2^5 = 32$ characters:

(1) $(+++++), (++++-), (+++-+), \ldots\ldots (+----), (-----)$;

the i^{th} sign of the character being plus or minus according as Σ does or does not satisfy the i^{th} postulate. The body of thirty-two propositions stating for the various characters represented in (1) that there exists or does not exist a system having the character in question constitutes what Professor Moore has called "the complete existential theory" of the five postulates.

For the five postulates in question the complete existential theory consists of fourteen propositions of existence and eighteen propositions of non-existence. The eighteen non-existencies arise from the fact that the negative of I implies III, IV, and V, and that also the negative of IV implies V.[6]

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[4]E. H. Moore, "Introduction to a Form of General Analysis," *New Haven Mathematical Colloquium*, Yale University Press, p. 82.

[5]J. S. Taylor, "Complete Existential Theory of Bernstein's Set of Four Postulates for Boolean Algebras," *Annals of Mathematics, Second Series*, vol. XIX, No. 1, pp. 64–69 (September, 1917).

[6]The question naturally arises as to whether it might not be possible to modify the postulates in a way such that they would become completely independent. The writer has investigated this possibility in considerable detail but has been unable to make such a modification without a considerable loss of simplicity. The simplest change found to bring about the desired results is as follows:—

I′. *K contains at least four distinct elements.*

V′. *There exists an element ϵ in K such that, for each a, b, c choice for which there exists any element in K satisfying the condition imposed upon 1 by the equation of IV for each pair of elements in the group of elements obtained by combining a, b, and c in all possible ways, the element ϵ does so,*

Propositions of Non-Existence

The eighteen propositions of non-existence, as implied above, may be expressed by the two following propositions:[7]

(2) $\Sigma^{-1} \supset \Sigma^{345}$

(3) $\Sigma^{-4} \supset \Sigma^{5}$

The truth of these two propositions is readily perceived from the following considerations. First, the hypothesis that I is not satisfied necessitates either K^{null} (a class without any elements) or $K^{singular}$ (a class with only one element). But if K contains no elements, postulates III, IV, and V are satisfied vacuously. And if K contains only one element, then they are satisfied either evidently or vacuously, according as Σ does or does not satisfy postulate II.

Secondly, the hypothesis that IV is not satisfied obviously results in the vacuous satisfaction of V.

Propositions (2) and (3) render impossible the existence of systems with the following eighteen characters.

$(-+++-)$, $(-++-+)$, $(-+-++)$, $(-++--)$, $(-+-+-)$,
$(-+--+)$, $(-+---)$, $(--++-)$, $(--+-+)$, $(---++)$,
$(--+--)$, $(---+-)$, $(----+)$, $(-----)$, $(+++--)$,
$(++---)$, $(+-+--)$, $(+----)$.

Propositions of Existence

The fourteen propositions of existence are established by the exhibition of fourteen systems having the remaining fourteen characters; there are two examples for $K^{singular}$, seven for K^{dual}, and five for K^{triple}. In each case K contains the least number of elements possible.

Examples for K singular

Systems having the characters $(-++++)$ and $(--+++)$ respectively are the following:

System I_1. Character $(-++++)$; class composed of single element e_1, with $e_1 - e_1 = e_1$.

System I_2. Character $(--+++)$; class composed of single element e_1, with $e_1 - e_1 \neq e_1$.

---

and such that, for such an a, b, c choice, if e' be defined as $\epsilon - e$, and if a, b, c, and the indicated combinations are elements of K,

$$a-(b-c) = [(a-b)' - (a-c')]'$$

Since considerable space would be occupied by a proof of the fact that the postulates thus modified are completely independent and since the reader should meet with no very serious difficulty in establishing this fact for himself, such proof is here omitted.

[7] $\Sigma^{-i} \supset \Sigma^{jh \cdots n}$ means, "If a system Σ does not satisfy the i^{th} postulate, then it does satisfy postulates j, k, ..., and n."

Examples for K dual

System II_1

$$(+++++),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & e_2 & e_1 \\ e_2 & e_2 & e_2 \end{array} \quad ;$$

System II_2

$$(+++-+),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & e_1 & e_1 \\ e_2 & e_2 & e_2 \end{array} \quad ;$$

System II_3

$$(+-+++),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & e_1 & e_1 \\ e_2 & e_2 & x \end{array} \quad ;$$

System II_4

$$(++-+-),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_1 & e_1 \end{array} \quad ;$$

System II_5

$$(++--+),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & e_1 & e_1 \\ e_2 & e_1 & e_1 \end{array} \quad ;$$

System II_6

$$(+-+-+),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & x & x \\ e_2 & x & x \end{array} \quad ;$$

System II_7

$$(+----+),\quad \begin{array}{c|cc} - & e_1 & e_2 \\ \hline e_1 & x & e_1 \\ e_2 & e_1 & e_1 \end{array}$$

Examples for K triple

System III_1

$$(++++-),\quad \begin{array}{c|ccc} - & e & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_2 & e_1 & e_3 \\ e_3 & e_3 & e_1 & e_1 \end{array} \quad ;$$

System III_2

$$(++-++),\quad \begin{array}{c|ccc} - & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e & e_1 \\ e_2 & e_3 & e_1 & e_2 \\ e_3 & e_1 & e_1 & e_1 \end{array} \quad ;$$

System III_3

$$(+-++-),\quad \begin{array}{c|ccc} - & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & x \\ e_2 & e_2 & e_1 & e_3 \\ e_3 & e_3 & e_1 & e_1 \end{array} \quad ;$$

System III_4

$$(+--++),\quad \begin{array}{c|ccc} - & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_1 & e_1 \\ e_2 & e_2 & e & e_2 \\ e_3 & e_1 & e_1 & x \end{array} \quad ;$$

System III_5

$$(+--+-),\quad \begin{array}{c|ccc} - & e & e_2 & e_3 \\ \hline e_1 & e_1 & e_2 & x \\ e_2 & e_1 & e_1 & x \\ e_3 & e_1 & e_2 & x \end{array}$$

III

THE ELEMENT 1 AND NEGATION

It is interesting to note that it is impossible to express either the element 1 or negation, "not$-$a," directly in terms of the operation of "$+$" or "$-$," although this can be done in terms of "rejection." It has been found necessary in all sets of postulates in terms of "logical addition" or "exception," therefore, to postulate one or both of these two ideas. Curiously enough, although there are several sets in which only 1 is postulated and "not$-$a" then defined, there has been no set formulated in which "not$-$a" is postulated and the element 1 defined. This might lead one to believe that the element 1 plays a more important rôle than negation, but that this does not follow in the case of "exception," at least, is demonstrated by the following set of five postulates in which only "not$-$a" is postulated.

I. K contains at least two distinct elements.

II. If a and b are elements of K, $a-b$ is an element of K.

III. If a, b, and the combinations indicated are elements of K, $a-(b-b)=a$

IV. For every element e in K there exists another element $e^{\cdot}$ in K, unique for each e, such that, if the combinations indicated are elements of K,

(1) $e-e^{\cdot}=e$

and (2) $a-(b-c)=[(c-a^{\cdot})^{\cdot}-(b^{\cdot}-a^{\cdot})]^{\cdot}$,

where each dotted element is an element satisfying (1).

Definition 1. $a^{\cdot\cdot}=(a^{\cdot})^{\cdot}$

Theorem 1. $a^{\cdot\cdot}=a$

Proof. Set $b=c=a$ in IV (2)

the left becomes $a-(a-a)=a$ by III,

while the right becomes $[(a-a^{\cdot})^{\cdot}-(a^{\cdot}-a^{\cdot})]^{\cdot}=[(a-a^{\cdot})^{\cdot}]^{\cdot}$ by III,

$=[(a)^{\cdot}]^{\cdot}$ by IV (1),

$=a^{\cdot\cdot}$ by Def. 1.

Theorem 2. $a^{\cdot}-a=a^{\cdot}$ by IV (1), Th. 1.

Theorem 3. $b^{\cdot}-a^{\cdot}=a-b$

Proof. Set $a=a$, $b=b$, $c=b^{\cdot}$ in IV (2)

$$a-(b-b^{\cdot})=[(b^{\cdot}-a^{\cdot})^{\cdot}-(b^{\cdot}-a^{\cdot})]^{\cdot}$$

whence $a-b=[(b^{\cdot}-a^{\cdot})^{\cdot}]^{\cdot}$ by IV (1), Th. 2.

$=b^{\cdot}-a^{\cdot}$

Corollary. $a^{\cdot}-b=b^{\cdot}-a$

Theorem 4. $a-(b-c)=[(b^{\cdot}-a^{\cdot})^{\cdot}-(c-a^{\cdot})]^{\cdot}$ by Th. 3, Cor.

Theorem 5. $(b^{\cdot}-a)^{\cdot}-(b-a)=a$

Proof. Set $a=a^{\cdot}$, $c=b$ in Th. 5.

Definition 2. $1=(a-a)^{\cdot}$

The rest of the development and the proof of the sufficiency of the set of postulates follow so closely those of the set for which it has already been explained in detail that the reader is left to complete the work for himself.

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